About one non linear recurrence of the third order
Sequences
https://www.linkedin.com/groups/8313943/8313943-6437179333101649921
Let k be an integer greater than 1. The sequence $(a_n)_{n\geq 0}$ is defined by
$a_0 = 4, a_1 = a_2 = \left(k^2-2\right)^2$ and
$a_{n+1} = a_n a_{n-1} - 2\left(a_n + a_{n-1}\right) - a_{n-2} + 8$ for $n \geq 2$.
Prove that $2+\sqrt{a_n}$ is a perfect square all $n \in \mathbb{N}$.
Solution by Arkady Alt, San Jose,California, USA.
Since
(1) $a_{n+2} = a_{n+1}a_n - 2\left(a_{n+1}+a_n\right) - a_{n-1} + 8 \iff$
$a_{n+2} - 2 = \left(a_{n+1}-2\right)\left(a_n - 2\right) - \left(a_{n-1}-2\right)$
then for $b_n := a_n - 2$ we obtain recurrence
(2) $b_{n+2} = b_{n+1}b_n - b_{n-1}, n \in \mathbb{N}$ with $b_0 = 2, b_1 = b_2 = \left(k^2-2\right)^2 - 2$.

Lemma.
Let sequense $(P_n)_{n\geq 0}$ be determined by the recurrence
(3) $P_{n+2} = P_{n+1}P_n - P_{n-1}, n \in \mathbb{N}$ with $P_0 = 2, P_1 = P_2 = x > 2$,
and let (f_n) be sequence of Fibonacci numbers ($f_{n+1} = f_n + f_{n-1}, n \in \mathbb{N}$ and $f_0 = 1,\ f_1 = 1$).
Then requrrence **(3)** determine polynomial $P_n(x)$ of x, of degree f_n with integer coefficients, such that $P_n(\cosh(t)) = 2\cosh(f_n t)$,
where $t := \cosh^{-1}\left(\dfrac{x}{2}\right)$

Proof.
Since $2\cosh(f_0 t) = 2\cosh(0) = 2, 2\cosh(f_1 t) = 2\cosh(f_2 t) =$
$2\cosh t = x$ and $2\cosh(f_{n+1}t)\cdot 2\cosh(f_n t) - 2\cosh(f_{n-t}t) =$
$4\cosh(f_{n+1}t)\cosh(f_n t) - 2\cosh(f_{n-t}t) =$
$2\left(\cosh(f_{n+1}t + f_n t) + \cosh(f_{n+1}t - f_n t)\right) - 2\cosh(f_{n-t}t) =$
$2\cosh(f_{n+2}t) + 2\cosh(f_{n-1}t) - 2\cosh(f_{n-t}t) = 2\cosh(f_{n+2}t)$
then by Math Induction we obtain
that $P_n(x) = 2\cosh\left(f_{n+2}\cdot\cosh^{-1}\left(\dfrac{x}{2}\right)\right)$.

Coming back to recurrence **(2)** and denoting $t := \cosh^{-1}\left(\dfrac{k}{2}\right)$ we obtain that
$\left(k^2-2\right)^2 - 2 = \left(4\cosh^2 t - 2\right)^2 - 2 = 4\left(2\cosh^2 t - 1\right)^2 - 2 = 4\cosh^2 2t - 2 =$
$2\left(2\cosh^2 2t - 1\right) = 2\cosh 4t$ and then accordingly to Lemma $b_n = 2\cosh(4f_n t)$.
Therefore, $a_n = 2\cosh(4f_n t) + 2 = 4\cosh^2(2f_n t)$ and since $\cosh(x) > 0$ for any x then $2+\sqrt{a_n} = 2 + 2\cosh(2f_n t) = 2\left(1+\cosh(2f_n t)\right) =$
$4\cosh^2(f_n t) = \left(2\cosh(f_n t)\right)^2 = \left(P_n(k)\right)^2$.